CAPM

Date 29/9/16
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MPT : Markowitz

CAPM : Sharpe

fundamental notions

1. Notion of Risk :

Riskiness is measured by the variability of returns.

Let's take an example to understand this.
5 years return given
Share M : 30 %, 28 %, 34 %, 32 % and 31 %.
Share N : 26 %, 13 % 48 % 11 % and 57 %.

Average Return :
Sh. M : $(30+28+34+32+31)/5 = 31\%$
Sh. N : $(26+13+48+11+57)/5 = 31\%$
So, which share would you choose to invest in?
You are at fix !!!
But, 2 things come to mind
(i) Expected Return (?)
(ii) Riskiness of Return. (?)

To a great extent, Sh M. shows steady return and Sh N shows great variability i.e. rise and fall is too sharp.

Hence, Sh. N is a risky asset; and investor would prefer Sh M to Sh. N.

2. **Notion of Diversification:**
Do not keep all your eggs in a single basket.

Different shares have different return patterns and hence, different variability.

Loss of ~~one~~ return of one share can be offset by the gain in other. So investor adds one more share into his kitty.

So diversification helps him reduce the variability in return.

3. **Notion of Portfolio:**

A set of all securities is portfolio in which we are (MPT) interested in.
(MPT: Modern Portfolio Theory)

MPT talks about a portfolio.

4. **Notion of dominance:**

(a) If two portfolios have identical expected returns, investor shall choose the portfolio with lower risk.

(b) If two portfolios have identical risk, he will choose the one having higher expected return.

5. Notion of Market Risk (Non-diversifiable)

Diversification reduces risk, yet a highly or well diversified portfolio is not free from variability in return (i.e. risk.)

Even if all securities traded in the Stock Exchange are taken to construct a portfolio, there is variability in return (risk).

This risk is known as Market Risk or non-diversifiable risk.

It is evident from the fluctuations in the market index.

6. Notion of Beta (β)

Market risk cannot be eliminated through diversification.

Therefore, we have to measure the risk of the security, with respect to its vulnerability to market ~~securit~~ risk.

If the market goes down by 1%, would the security go down by 0.5%, 1% or 2%?

This sensitivity of security to the movements of the market is known as the Beta Coefficient of the security.

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7. Notion of Trade off between Risk and Return

If the securities are correctly priced, the return on each security would be commensurate with its risk measured by its Beta (β).

(a) The graphical depiction of the resulting straight line relationship between return and beta (β) is known as SML.

(b) The straight line relationship between return and risk (σ) of a well diversified portfolio is called Capital Market Line (CML.)

Measurement of Risk and Return in individual stocks

Return indicates 2 components
(a) Capital appreciation
(b) Dividend

So, Rate of Return (r) $= \frac{P_1 - P_0}{P_0} + \frac{D_1}{P_0}$

Let's find return using the formula as above:
Stock's name : MRF Ltd.

Year	Price (Rs)	Dividend (Rs)	Rate of Return (r)	(1+r)
2006	115	-		-
2007	115	12	0.1043	1.1043
2008	190	15	0.7826	1.7826
2009	295	15	0.6316	1.6316
2010	315	15	0.1186	1.1186
2011	230	15	(−) 0.2222	0.7778
2012	260	15	0.1957	1.1957
2013	230	15	−0.0577	0.9423
2014	385	25	0.7826	1.7826
2015	735	25	0.9740	1.9740
2016	1065	40	0.5034	1.5034

$\Sigma r = 3.813$

Average Return = $\frac{3.813}{10}$ = 0.3813

in % = 38.13 %

How do we find compound rate of growth of return (r) ?

Let's find the geometric mean (GM)

$$r_g = \sqrt[n]{\prod_{t=1}^{n} (1+r_t)} - 1$$

In the above example:

$$r_g = \sqrt[10]{16.6565} - 1$$

$$= 1.3250 - 1$$

$$= .3250$$

or in % = 32.50%

How do we get 16.6565 ?

Take it easy : simple : multiply all (1+r)

$\Rightarrow$ (1.1043)(1.7826)(1.6316)(1.1186)(0.7778)(1.7826) x (1.1957)(1.9740)(1.5034)(0.9423)

= 16.6565 (Got it ?)

Let's find the measure of risk

It is the standard deviation of return; statistically we denote it as σ (sigma)

~~Remember~~ Remember? The formula:

$$\sigma = \sqrt{\frac{\sum(r-\bar{r})^2}{n-1}}$$ ∴ what is the variance?

Let's find the σ in the example given above:

Year	Return	$(r-\bar{r})$	$(r-\bar{r})^2$
2007	0.1043	−0.2770	0.0767
2008	0.7826	0.4013	0.1610
2009	0.6316	0.2503	0.0626
2010	0.1186	−0.2627	0.0690
2011	−0.2222	−0.6035	0.3642
2012	0.1957	−0.1856	0.0345
2013	−0.0577	−0.4390	0.1927
2014	0.7826	0.4013	0.1610
2015	0.9740	0.5927	0.3513
2016	0.5043	0.1221	0.0149
	$\sum r = 3.813$		1.4881

$\bar{r} = 3.813/10 = 0.3813$

$r-\bar{r} = 0.1043 - 0.3813 = -0.2770$ ... find so on.

variance $(\sigma^2) = \frac{\sum(r-\bar{r})^2}{n-1} = \frac{1.4881}{10-1} = \frac{1.4881}{9} = 0.1653$

$\therefore \sigma = \sqrt{0.1653} = 0.4066$

In our example at page 5
How do we get the return?
Let me work out first two.

For 2007 : $r = \dfrac{P_1 - P_0}{P_0} + \dfrac{D}{P_0}$

$= \dfrac{115-115}{115} + \dfrac{12}{115}$

$= 0 + .1043$

$= .1043$

For 2008 : $r = \dfrac{190-115}{115} + \dfrac{15}{115}$

$= \dfrac{75}{115} + \dfrac{15}{115} = \dfrac{90}{115} = 0.7826$

For 2009 : $r = \dfrac{295-190}{190} + \dfrac{15}{190}$

$= \dfrac{105}{190} + \dfrac{15}{190} = \dfrac{120}{190} = 0.631$

Measurement of Risk and Return of Portfolio

Harry Markowitz developed Portfolio theory.

He highlighted how diversification reduces the risk of a portfolio without affecting rate of return.

We will now learn how to find the risk of a portfolio and its return through examples in this section.

First we'll start with two securities.

Measuring Portfolio returns:

Portfolio Return = Weighted average of returns

$$r_{pt} = \sum_{t=1}^{n} r_{it} \, w_{it}$$

OR

$$r_p = r_A w_A + r_B w_B$$

r_{pt} = Return from the portfolio for 't' time

r_{it} = Return of ith security for 't' time period

w_{it} = weight of ith asset at the beginning of time period t

Let's solve a problem:

Eg: A portfolio consists of 2 equities A and B. At the beginning of the year the market value stock A was Rs 120 and that of B was Rs 80. During the year both the equities paid a dividend of Rs 2 each and at the end of the year stock A was quoting at Rs 132 and B was quoting at Rs 96.

Compute the return of the Portfolio

Ans: Return of stocks A and B:

$$r_A = \frac{D}{P_0} + \frac{(P_1 - P_0)}{P_0} = \frac{2 + (132 - 120)}{120} = \frac{14}{120}$$

$$= 0.1167 \text{ or } 11.67\%$$

$$r_B = \frac{2 + 96 - 80}{80} = \frac{18}{80} = 0.225 = 22.50\%$$

Portfolio details are:

Market Value:

At the beginning = Rs 120 + Rs 80 = Rs 200

At the end = Rs 132 + Rs 96 = Rs 228

Dividend Received: Rs 2 + Rs 2 = Rs 4.00

Portfolio return =

$$r_p = \frac{D_{pt} + (P_{pt} - P_{pt-1})}{(P_{pt-1})}$$

$$= \frac{4 + (228 - 200)}{200}$$

$$= 16\%$$

Otherwise:

$$r_p = r_A w_A + r_B w_B$$

$$= (11.67\%)\frac{120}{200} + (22.5\%)\frac{80}{200}$$

$$= 7\% + 9\%$$

$$= 16\%$$

Measuring Portfolio Risks: (σ_p)

Measuring portfolio risk is not that easy. (why?)

(i) The portfolio consists of more than one security.

(ii) The inter-relationship between the securities may be negative, positive or neutral. (Essentially, we are talking about the co-relation)

(iii) The loss in one security may offset the gain in the other. It means; risk can be reduced due to diversification.

(iv) The variability of portfolio return σ_p depends on the variability of security return (σ_x) or (σ_y) etc and also on the corelation coefficient (r_{xy}) or (ρ_{xy}) ~~spell~~ pronounce as rho xy.

Let's start ~~the~~ with an example:

Expected Return on Security X = 20%
Expected Return on Security Y = 30%
σ of Security X = 10%
σ of Security Y = 16%

Let's invest 40% in X and 60% in Y.

Now, there are 3 scenarios:

say; $\rho_{xy} = -1, 0.5, \text{or } +1$

ρ_{xy} = Co-relation Coefficient between X and Y

Let's evaluate the impact on the gains from diversification, for 3 different values of correlation coefficient.

$$\text{Expected Return of portfolio} = \frac{20}{100} \times \frac{40}{100} + \frac{30}{100} \times \frac{60}{100}$$

$$= 8\% + 18\%$$

$$= 26\%$$

What is the portfolio Risk? (σ_p)

Can it be? $\frac{10}{100} \times \frac{40}{100} + \frac{16}{100} \times \frac{60}{100}$

$$= 4\% + 9.6\% = 13.6\%$$

Ans: It depends.

Case-I : If the correlation coefficient is +1, then, it would be 13.6 %.

Case-II : if the correlation coefficient is less than +1 then, σ_p would be less than 13.6 %.

If it is 0.5, The σ_p = 12.11 %.

Case III : if the ρ = −1.0, then σ_p = 5.6 %.

[We will compute shortly, Pl. wait]

Some food for thought:

i) If we invest all our money in X, our security return is 20% for a risk of 10%.

2) If we invest in Y security alone, the return = 30% and the risk is 16%.

Thus, we will take an additional risk of 6% for a rise of return from 20% to 30%.

It means 6% additional risk would fetch us 10% additional gain.

or; 1% additional gain means 0.6% additional risk.

3) If we invest in both X and Y and correlation coefficient is +1, the $\sigma_p = 13.6\%$.

So no gain in diversification.

4.) If the return rose from 20% to 26% the risk goes up from 10% to 12.1%.

It means, for every 1% rise in return there is an increase in risk of 0.35%.

i.e. $2.1\% \div 6\%$

$= 2\frac{1}{10} \div 6\% = \frac{21}{10} \div 6 = \frac{21}{10} \times \frac{1}{6} = \frac{7}{20}$

$= \frac{7}{20} \times 100 = 0.35\%$.

Let us see the portfolio Returns and Risk based on proportion of investment and correlation coefficient.

Proportion Invested in X $r=20\%$	Proportion Invested in Y $r=30\%$	Expected Return r_p	value of σ when Coefficient Correlation $=1.00$	$=0.5$	$=-1.0$
1.00	0.0	20	10	10	10
0.90	0.10	21	10.6	9.90	7.40
0.80	0.20	22	11.2	9.99	4.80
0.75	0.25	22.50	11.50	10.11	3.50
0.615	0.385	23.85	12.31	10.66	0.00
.60	0.40	24.00	12.40	10.74	0.40
.50	0.50	25.0	13.00	11.36	3.00
.40	0.60	26.0	13.6	12.11	5.60
.20	0.80	28.0	14.8	13.91	10.80
.10	0.90	29.0	15.4	14.93	13.40
0.0	1.00	30.0	16	16	16

For a 2 asset case Portfolio

How do we calculate σ_p?

We know $(a+b)^2 = a^2 + b^2 + 2ab$

This formula is extended to calculate σ_p

See the matrix Table below:

		W_A	W_B
		A	B
W_A	A	σ_A^2	COV (AB)
W_B	B	$COV_{(AB)}$	σ_B^2

$$\sigma_p = \sqrt{W_A \cdot W_A \cdot \sigma_A^2 + W_A W_B \cdot COV_{(AB)} + W_A \cdot W_B \cdot COV_{(AB)} + W_B \cdot W_B \cdot \sigma_B^2}$$

$$= W_A^2 \sigma_A^2 + W_A \cdot W_B \cdot COV_{AB} + W_A \cdot W_B\, COV_{(AB)} + W_B^2 \sigma_B^2$$

$$= W_A^2 \sigma_A^2 + 2\, W_A W_B \cdot COV_{AB} + W_B^2 \sigma_B^2$$

where, W_A : Weightage of investment in Sec A.

W_B : Weightage of investment in Security B

σ_A : Risk of ~~port~~ Security A (St. Dev)

σ_B : Std. dev. of Security B

cov_{AB} = Covariance of A & B

$= r_{AB}\,\sigma_A\,\sigma_B$

$$\boxed{\text{We know; } r_{AB} = \frac{cov(AB)}{\sigma_A \cdot \sigma_B}}$$

For a 3 asset case Portfolio

		W_A	W_B	W_C
		A	B	C
W_A	A	σ_A^2	$cov(AB)$	$cov(AC)$
W_B	B	$cov(AB)$	σ_B^2	$cov(BC)$
W_C	C	$cov_{(AC)}$	$cov_{(BC)}$	σ_C^2

Portfolio Risk $= \sqrt{\begin{aligned} & w_A \cdot w_A \cdot \sigma_A^2 + w_A w_B\, cov_{AB} + w_A w_C\, cov_{AC} \\ & + w_B \cdot w_A \cdot cov_{AB} + w_B w_B \sigma_B^2 + w_B w_C\, cov_{BC} \\ & + w_A \cdot w_C\, cov_{AC} + w_B w_C\, cov_{B.C} + w_C \cdot w_C\, \sigma_C^2 \end{aligned}}$

$= w_A^2 \sigma_A^2 + 2 w_A \cdot w_B \cdot cov_{(AB)} + 2 w_A \cdot w_C \cdot cov_{(AC)} + 2 w_B w_C\, cov_{(BC)}$

$+ w_B^2 \sigma_B^2 + w_C^2 \cdot \sigma_C^2$

Let's solve problems to find r_p and σ_p

Q1: 2-asset case:

Following table shows the returns of security X and Y in a portfolio. Find,
(i) Coefficient of Correlation
(ii) Covariance
(iii) Portfolio Return
(iv) Portfolio risk.

Year	Return X	Return Y
2010	40	-10
2011	-10	40
2012	35	5
2013	-5	35
2014	15	15

Ans:

Year	R_x	R_y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})(y-\bar{y})$
2010	40	-10	25	-25	-625
11	-10	40	-25	25	-625
12	35	-5	20	-20	-400
13	-5	35	-20	20	-400
14	15	15	0	0	0
	$\sum R_x = 75$	$\sum R_y = 75$			$\sum = -2050$

$\therefore \bar{x} = \frac{75}{5} = 15 \quad \bar{y} = \frac{75}{5} = 15$

(ii) $cov_{xy} = \cancel{\Sigma 2050}$

$$\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$$

$$= \frac{-2050}{5}$$

$$= -410$$

(i) Correlation Coefficient $= \dfrac{cov_{xy}}{\sigma_x \sigma_y}$

You have already known how to find σ_x & σ_y

So, $\sigma_x = \cancel{22.6} = \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{2050}{5}} = \sqrt{410}$

$= 20.25$

$\sigma_y = \cancel{22.6} = \sqrt{\dfrac{\Sigma(y-\bar{y})}{n}} = \dfrac{-2050}{5} = \sqrt{-410} = 20.25$

Now. $CC = \dfrac{-410}{\cancel{22.6 \times 22.25}\ 20.25 \times 20.25} = -1$

(iii) Portfolio Return:
Suppose, we invest our money equally in securities x and y

$\therefore w_x = 0.5 \quad w_y = 0.5$

$r_p = w_x \cdot r_x + w_y \cdot r_y$

$= (0.5)(15) + 0.5(15)$

$= 7.5 + 7.5$

$= 15\%$

(iv) Portfolio Risk $\sigma_p = ?$ 0

$\sigma_p = \cancel{w_x^2 \sigma_x^2}$

$= \sqrt{w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y \, cov_{xy}}$

$= \sqrt{\cancel{(0.5)^2 (0.226)^2 + (0.5)^2 (0.226)^2 + 2(0.5)(0.5)(-410)}}$

$= \sqrt{\left(\frac{1}{2}\right)^2 (20.25)^2 + \left(\frac{1}{2}\right)^2 (20.25)^2 + 2 \times \frac{1}{2} \times \frac{1}{2} (-410)}$

$= \sqrt{\frac{1}{4} \times 820 + \frac{1}{2}(-410)}$

$= \sqrt{205 - 205} = 0$

Note:

We see $r = -1$
Therefore, σ_p becomes zero (0), meaning there is no risk at all.

[In real life impossible]

Home Task:

The portfolio contains 2 assets: namely A and B. 2/3 is invested in A and rest in B. Expected Returns and Risks of these securities are given below:

Share	ER(%)	σ %)
A	5	20
B	15	40

Find; Return of Portfolio r_p and Risk of Portfolio σ_p

$$E_{R_p} = w_a r_a + w_b r_b$$

$$= \frac{2}{3} \times 5 + \frac{1}{3}(15)$$

$$= 3.33 + 5 = 8.33\ \%$$

$$\sigma_p = \sqrt{w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a \cdot w_b\, COV_{AB}}$$

$$= \sqrt{w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \cdot r_{ab}\, \sigma_a \sigma_b}$$

$$= \sqrt{\left(\frac{2}{3}\right)^2 (20)^2 + \left(\frac{1}{3}\right)^2 (40)^2 + 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(20)(40) \cdot r_{AB}}$$

$$= \sqrt{\frac{4}{9} \times 400 + \frac{1}{9} \times 40 \times 40 + 2 \cdot \frac{2}{9} \cdot 800\, r_{AB}}$$

$$= \sqrt{\frac{1600}{9} + \frac{1600}{9} + 2 \times \frac{2}{9} \cdot 800 \cdot r_{AB}}$$

$$= \sqrt{\frac{3200}{9} + \frac{3200}{9} r_{AB}}$$

Suppose $r_{AB} = 1$, $\sigma_p = \sqrt{\frac{6400}{9}} = \frac{80}{3}$

$$= 26.67\ \%$$

If $r_{AB} = 0$; $\sigma_p = \sqrt{\frac{3200}{9} + \frac{3200}{9}(0)}$

$$= \sqrt{\frac{3200}{9}} = \frac{56.57}{3}$$

$$= 18.86\ \%$$

if $r_{AB} = -1$; $\sigma_p = \sqrt{\frac{3200}{9} + \frac{3200}{9}(r_{AB})}$

$$= \sqrt{\frac{3200}{9} + \frac{3200}{9}(-1)}$$

$$= \sqrt{\frac{3200}{9} - \frac{3200}{9}}$$

$$= \sqrt{0} = 0$$

Q A portfolio consists of four securities and the proportion of investment and individual returns are as follows:

Security	Returns (%)	Proportion of Inv.
A	12	0.2
B	17	0.3
C	23	0.1
D	20	0.4

Calculate the portfolio return:

Ans.

$$r_p = w_a r_a + w_b r_b + w_c r_c + w_d r_d$$

$$= (0.2)(12) + (0.3)(17) + (0.1)(23) + (0.4)(20)$$

$$= 2.4 + 5.1 + 2.3 + 8.0$$

$$= 17.8 \%$$

Q The variance and co-variance matrix is given below. Find the portfolio risk.

wt →		0.2	0.3	0.5
↓		A	B	C
0.2	A	52	63	36
0.3	B	63	38	74
0.5	C	36	74	45

Ans: This is a portfolio consisting of 3 Assets. Looking at the matrix in the question and mapping it in the matrix, we get

$$\sigma_A^2 = 52 \quad \sigma_B^2 = 38 \quad \sigma_C^2 = 45$$

$cov_{AB} = 63$

$cov_{AC} = 36$ [See formula at page 17]

$cov_{BC} = 74$

$$\therefore \sigma_p^2 = (0.2)(0.2)\sigma_A^2 + (0.2)(0.3)\,cov_{AB} + (0.2)(0.5)\,cov_{AC}$$
$$+ (0.3)(0.2)(cov_{AB}) + (0.3)(0.3)(\sigma_B^2) + (0.3)(0.5)\,cov_{BC}$$
$$+ (0.5)(0.2)(cov_{AC}) + (0.5)(0.3)(cov_{BC}) + (0.5)^2(\sigma_C^2)$$

$$= (0.2)^2(52) + (0.2)(0.3)(63) + (0.2)(0.5)(36) +$$
$$(0.3)(0.2)(63) + (0.3)^2(38) + (0.3)(0.5)(74)$$
$$+ (0.5)(0.2)(36) + (0.5)(0.3)(74) + (0.5)^2(45)$$

$$= 53.71$$

$$\therefore \sigma_p = \sqrt{53.71} = 7.33 \text{ Ans}$$

How do you compute σ_p when correlation efficients are given?

See the example below:

Q: Security wt. σ_i Correlation Coefficient

Security	wt.	σ_i	Correlation Coefficient
P	0.35	7	PQ : 0.7
Q	0.25	16	PR : 0.3
R	0.40	9	QR : 0.4

Compute σ_p ?

we know; $Cov_{ab} = Cor.Coeff_{ab} \cdot \sigma_a \sigma_b$

∴ Applying the formula; we get

$Cov_{PQ} = (0.7)(7)(16) = 78.4$

$Cov_{PR} = (0.3)(7)(9) = 18.9$

$Cov_{QR} = (0.4)(16)(9) = 57.6$

Let's put the data in the covariance matrix

wt.		P 0.35	Q 0.25	R 0.40
0.35	P	49	78.4	18.9
0.25	Q	78.4	256	57.6
0.40	R	18.9	57.6	81

Putting the value, we get :

$$\sigma_p^2 = (0.35)^2(49) + (.35)(.25)(78.4) + (.35)(.40)(18.9)$$

$$+ (0.25)(0.35)(78.4) + (0.25)^2(256) + (0.25)(0.40)(57.6)$$

$$+ (0.4)^2(0.35)(18.9) + (0.4)(0.25)(57.6) + (0.4)^2(81)$$

$$= 65.4945$$

$$\therefore \sigma_p = \sqrt{65.4945}$$

$$= 8.09$$

Q Home task :

Find E_{Rp} and σ_p^2 of a portfolio comprising two securities, assuming that the portfolio weight are 0.75 for A and 0.25 for B. The expected return for security A is 18% and its standard deviation σ_A is 12% while the expected return and standard deviation (σ_B) for B are 22% and 20% respectively. The correlation between A and B is 0.6

Q: The historical rates of return of 2 securities over the past 10 yrs are given below. Calculate the covariance and the correlation of the 2 securities.

	1	2	3	4	5	6	7	8	9	10
A	12	8	7	14	16	15	18	20	16	22
B	20	22	24	18	15	20	24	25	22	20

Calculation of covariance:

Year	R_A	Deviation(A) $(R_A - \bar{R}_A)$	R_B	Dev.(B) $(R_B - \bar{R}_B)$	(Dev A) × Dev. B
1	12	−2.8	20	−1	2.8
2	8	−6.8	22	1	−6.8
3	7	−7.8	24	3	−23.4
4	14	−0.8	18	−3	2.4
5	16	1.2	15	−6	−7.2
6	15	0.2	20	−1	−0.2
7	18	3.2	24	3	9.6
8	20	5.2	25	4	20.8
9	16	1.2	22	1	1.2
10	22	7.2	20	−1	−7.2

$\bar{R}_A = \frac{148}{10}$ $\quad$ $\bar{R}_B = \frac{210}{10}$ $\quad$ $\Sigma = -8.0$

$= 14.8$ $\quad$ $= 21$

$$cov_{AB} = \frac{\Sigma(R_A - \bar{R}_A)(R_B - \bar{R}_B)}{n}$$

$$= \frac{-8.0}{10}$$

$$= -0.8$$

In order to calculate correlation, we have to find the standard deviations of 2 securities.

Year	R_A	$R_A - \bar{R}_A$	$(R_A - \bar{R}_A)^2$
1	12	−2.8	7.84
2	8	−6.8	46.24
3	7	−7.8	60.84
4	14	−0.8	0.64
5	16	1.2	1.44
6	15	0.2	0.04
7	18	3.2	10.24
8	20	5.2	27.04
9	16	1.2	1.44
10	22	7.2	51.84
Σ =	148		Σ = 207.60

$\therefore \bar{R}_A = \frac{148}{10}$

$= 14.8$

$\sigma_A = \sqrt{\frac{\Sigma(R_A - \bar{R}_A)^2}{n}}$

$= \sqrt{\frac{207.60}{10}}$

$= \sqrt{20.76}$

$= 4.56$

Similarly, $\sigma_B = \cancel{2.8678}$ 2.89 (Pl. do it)

Now Corr. Coeff$_{AB}$ $= \frac{COV_{AB}}{\sigma_A \sigma_B} = \frac{-0.8}{4.56 \times \cancel{2.8678}\ 2.89}$

$= \cancel{-0.0061}$ −0.06 Ans

finding Expected Return and Risk based on probability of occurence.

Q

Estimated probability Distribution

State of Economy	Probability (p)	Rate of Return if given state occurs: Electronics %	Whitegoods %
Strong Boom (SB)	0·10	36	32
Mild Boom (MB)	0·20	25	28
Av. Economy (AE)	0·40	18	22
Mild Recession (MR)	0·20	10	14
Strong Recession (SR)	0·10	−3	5

Ans:

$$ER = \sum R_j P_j$$

Calculation Table

Event	Prob (P_j)	Return on Electronic (R_j)	($P_j R_j$)	Return on Whitegoods (R_j)	P ($P_j R_j$)
SB	0·10	36	3·6	32	3·2
MB	0·20	25	5·0	28	5·6
AE	0·40	18	7·2	22	8·8
MR	0·20	10	2·0	14	2·8
SR	0·10	−3	−0·3	5	~~0·5~~ (0·5)
			17·5 %		20.9 %

Risk on Electronic Scrip:

$$\sigma = \sqrt{\sum (R_j - \bar{R}_j)^2 (P_j)}$$

Possible	Return	Probability (P_j)	$(R_j - \bar{R}_j)$	$(R_j - \bar{R}_j)^2$	$(R_j - \bar{R}_j)^2 (P_j)$
SB	36	0.10	18.5	342.25	34.225
MB	25	0.20	7.5	56.25	11.250
AE	18	0.40	0.5	0.25	0.100
MR	10	0.20	−7.5	56.25	11.250
SR	−3	0.10	−20.5	420.25	42.025
				875.25	98.850

~~$\sum = 86$~~

$$\bar{R}_j = (36 \times .10) + (25 \times 0.20) + (18 \times .4) + (10 \times .2) + (-3)(0.10)$$

$$= 3.6 + 5.0 + 7.2 + 2 - .3$$

$$= 17.5$$

$$\therefore \sigma = \sqrt{\sum (R_j - \bar{R}_j)^2 (P_j)}$$

$$= \sqrt{98.85}$$

$$= 9.94 \%$$

Home task: Can you calculate σ value for white goods?

Total Risk

Total Risk = Diversifiable Risk + Undiversifiable Risk

Diversifiable Risk :

* It is unique to the firm
* The sources of unsystematic risk (i.e. diversifiable risk) are, events like;
  - Labour strike
  - Errors of judgement by management
  - Inventions by employees
  - Strong and effective advt. campaigns.
  - Shift in demand of Co's products.
  - Legal cases against the company.

* It affects the firm only, thus its impacts on the price movement is felt by the company and hence on the return; and the variability in return (σ).

* The σ (risk) of the company's return can be averaged to zero by adding security of other company.

Mathematically;

$$r = E_{(r)} + e_i$$

Undiversifiable Risk (Systematic)

* It is caused by market factors that affect all security prices simultaneously.

* Factors like
  - changes in economic, social and political environment
  affect prices of all securities at the same time.

* Most securities tend to move together in a systematic manner as all shares are positively correlated.

* Mathematically,

$$E_{r_i} = a_i + b_i (E)_{r_m}$$

E_{r_i} = Expected return from the ith security which is a linear function of expected return of the market.

a_i = Alpha constant; a value close to '0'

b_i = Beta, a measure of undiversifiable risk

'Beta' measures the change that can be expected in the return from a share as a result of a unit change in the market return.

Q Quarterly Data on Market Returns and Return of a share W :

		Return on market (NIFTY-50)	Return on Sh W
2015 / Q1		9.2	7.4
	Q2	7.3	8.5
	Q3	−4.1	−3.4
	Q4	17.2	13.7
2016	Q1	14.0	14.5
	Q2	5.9	3.2
	Q3	−6.9	−0.9
	Q4	5.6	2.6

An

when these are plotted on a graph, we don't find a straight line. So, we have to find a line of best fit. This is otherwise known as line of least squares.

So, we have to determine some other statistics

Sl no.	Date	r_m	r_i	r_m^2	r_i^2	$r_m r_i$
1	15 Q_1	9.2	7.4	84.64	54.76	68.08
2	Q_2	7.3	8.5	53.29	72.25	62.05
3	Q_3	−4.1	−3.4	16.81	11.56	13.94
4	Q_4	17.2	13.7	295.84	187.69	235.64
5	16 Q_1	14.0	14.5	196 ~~37.81~~	210.25	203
6	Q_2	5.9	3.2	34.81 ~~47.61~~	10.24	18.88
7	Q_3	−6.9	−0.9	47.61	0.81	6.21
8	Q_4	5.6	2.6	31.36	6.76	14.56
	$\Sigma =$	48.2	45.6	760.36	554.32	622.36

$$E_{r_m} = \frac{48.2}{8} = 6.025 \qquad E_{r_i} = \frac{45.6}{8} = 5.7$$

We find; $\sigma_{(r_m)} = 7.66$ (Will you please!)

$\sigma_{r_i} = 6.07$

$$\text{Beta}(\beta) = \frac{n \times \Sigma r_m r_i - \Sigma r_m \times \Sigma r_i}{n \Sigma r_m^2 - (\Sigma r_m)^2}$$

$$= \frac{8(622.36) - (48.2)(45.6)}{8 \times 760.36 - (48.2)^2}$$

$$= 0.74$$

$$\alpha\,(\text{Alpha}) = E_{r_i} - b_i E_{r_M}$$

$$= 5.7 - (0.74)(6.025)$$

$$= 0.0124 = 1.24\,\%$$

$$\rho\,(\text{rho}) = \frac{n(\cancel{r_m})(r_i) - (\Sigma r_m)(\Sigma r_i)}{\sqrt{n(r_m^2) - (\Sigma r_m)^2}\sqrt{n(r_i)^2 - (\Sigma r_i)^2}}$$

$$= \frac{8(622.36) - (48.2)(45.6)}{\sqrt{8(760.36 - (48.2)^2}\sqrt{8(554.32 - (45.6)(45.6)}}$$

$$= 0.934$$

Now, we use regression equation to find coordinates x and y. $\boxed{Y = \alpha + \beta X}$

And a straight line could be drawn on the graph.

Q1. Calculate the E_R and σ for a stock having the following probability distributions of returns:

Possible returns	Probability of occurence
-25	0.05
-10	0.10
0	0.10
15	0.15
20	0.25
30	0.20
35	0.15

Ans: $E_R = 16.25$ and $\sigma = 16.57$

Q2 A stock costing Rs 120 pays no dividend. The possible prices that the stock might sell for at the end of the year with respective probabilities as follows:

Price (Rs)	Probability
115	0.1
120	0.1
125	0.2
130	0.3
135	0.2
140	0.1

Calculate expected return and standard deviation.

Ans: $E_R = 7.08\%$, $\sigma = 5.91\%$

[See pages 63-66 of the book by S. Kevin]

Q_3 An investor has analysed a share for a one-year holding period. The share is currentely selling for Rs 43 but pays no dividend and there is a fifty-fifty chance that the share will sell for either Rs 55 or Rs 60 by the year end. What is the expected return and risk if 250 shares are acquired with 80 percent borrowed funds? Assume the cost of borrowed funds to be 12%.

Ans:

Calculation of Returns:

Current price = Rs 43

if yr. end price = Rs 55, then

Probable return $= \frac{Rs\ 55 - Rs\ 43}{Rs\ 43} \times 100$

$= 27.91$

if yr. end price = Rs 60, then

Probable return $= \frac{Rs\ 60 - Rs\ 43}{43} \times 100$

$= 39.53$

Calculation of expected Return

(R)	(p)	(R)(p)
27.91	0.50	13.955
39.53	0.50	19.765
	$\bar{X}$ =	33.720

Calculation of Standard deviation:

X_i	$(X_1 - \bar{X})$	$(X_1 - \bar{X})^2$	p	$p(X_1 - \bar{X}_1)^2$
27.91	-5.81	33.76	0.5	16.88
39.53	5.81	33.76	0.5	16.88
			Variance =	33.76

$\therefore \sigma = \sqrt{33.76} = 5.81$

Return and Risk for buying 250 shares

Investment in 250 shares = Rs 43 × 250
= Rs 10,750

Borrowed funds (80%) = Rs 8600

Expected Return from 250 shares:

Gross Return $= \frac{10,750 \times 33.72}{100}$

= Rs 3624.90

Int to be paid = Rs $8600 \times \frac{12}{100}$ = Rs 1032.00

$\therefore$ Net Return = Rs 3624.90 − 1032 = Rs 2592.90

Risk in 250 shares:

Rs $10750 \times \frac{5.81}{100}$ = Rs 624.58

formulae

(i) correlation coefficient (ρ)

$$= \frac{n \sum XY - (\sum X)(\sum Y)}{\sqrt{n \sum X^2 - (\sum X)^2} \cdot \sqrt{n \sum Y^2 - (\sum Y)^2}}$$

(ii) Standard deviation σ

$$\sigma = \frac{\sqrt{N \sum X^2 - (\sum X)^2}}{N}$$

i.e, when we get R_i and N for i^{th} security

$$\frac{\sqrt{N \sum R_i^2 - (\sum R_i)^2}}{N}$$

for market st. deviation (risk)

$$\cancel{R_i} \; \frac{\sqrt{N(\sum R_m^2) - (\sum R_m)^2}}{n}$$

(iii) $$\beta_i = \frac{r_{im} \, \sigma_i \, \sigma_m}{\sigma_m^2}$$

r_{im} : coefficient of correlation between the ith security and market returns.

(IV) By using regression model,

$$\beta = \frac{n \sum XY - (\sum X)(\sum Y)}{n \cdot \sum X^2 - (\sum X)^2}$$

and $\alpha = \bar{Y} - \beta \bar{X}$

Q Monthly return data for ONGC stock and NSE index given below:

month	ONGC	NSE index
1	-0.75	-0.35
2	5.45	-0.49
3	-3.05	-1.03
4	3.41	1.64
5	9.13	6.67
6	2.36	1.13
7	-0.42	0.72
8	5.51	0.84
9	6.80	4.05
10	2.60	1.21
11	-3.81	0.29
12	-1.91	-1.96

(i) Calculate α and β

(ii) If NSE index moves up by 15% next month, how much return do you expect from ONGC stock. ?

PORTFOLIO SELECTION

Optimal portfolio: A portfolio that provides the highest return and lowest risk.

Portfolio Selection: The process of finding an optimal folio is portfolio selection.

Harry Markowitz is the pioneer of developing modern portfolio theory. 1952, 1959

Portfolio opportunity set: It is also known as portfolio feasible set. An investor can create a number of portfolios by combining a small number of securities in different proportions.

Each portfolio will have an expected return and a corresponding risk. When we arrange the same and put it in a table we find a large number of feasible set.

But, the investor would follow the/keep the principles of dominance in his mind and discard the inefficient portfolios. He is thus interested in efficient portfolios.

Let's go to find an efficient set of portfolios through the example.

Portfolio No	Expected Return %	Standard Deviation Risk (%)
1	5.6	4.5
2	7.8	5.8
3	9.2	7.6
4	10.5	8.1
5	11.7	8.1
6	12.4	9.3
7	13.5	9.5
8	13.5	11.3
9	15.2	12.7
10	16.8	12.9

Let's compare 4 and 5 portfolios:

for same risk level of 8.1, we have 2 portfolios with returns 10.5 and 11.7.
The portfolio 5 is more efficient than 4
why? because, the expected return is more in portfolio 5
So, the investor will discard portfolio No. 4 as portfolio no 5 dominates portfolio no 4.

Let's Compare 7 and 8

Here, expected returns are same (13.5) with a different levels of risk.

Obviously, investor will choose portfolio 7 and discard portfolio 8. (why?)

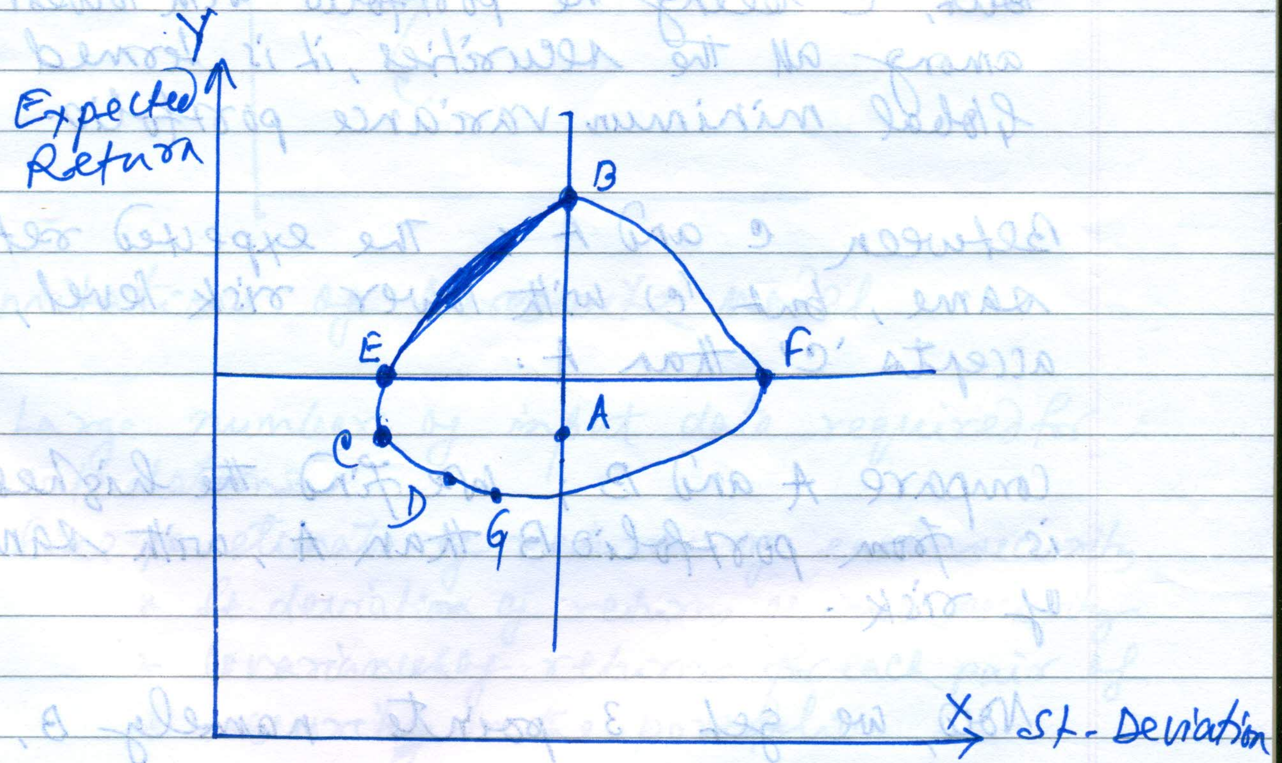
An investor behaves rationally.

So, the guiding criteria :

(i) With same expected return, the investor would choose with lower risk.

(ii) With the same level of risk, the investor would choose the one with higher return.

Let's us now graphically understand other concepts such as
(i) Efficient frontier.
(ii) Global minimum variance portfolio



The boundary shows the numerous portfolios showing risk and expected returns plotted in a graph.

Now, we consider F and E :
Both have same expected return.
Portfolio E has a lower risk

So, we accept portfolio E

Now, the investor will move upwards to find the opportunity of higher return and towards left to decrease his risk further.

Compare E and C : C is towards left of E and with lower risk ; but its return is too low.
But, C being the portfolio with lowest risk among all the securities, it is termed as Global minimum variance portfolio.

Between C and A ; The expected returns are same, but 'C' with lower risk level, so the accepts 'C' than A.

Compare A and B ; we find the highest return is from portfolio B than A, with same level of risk.

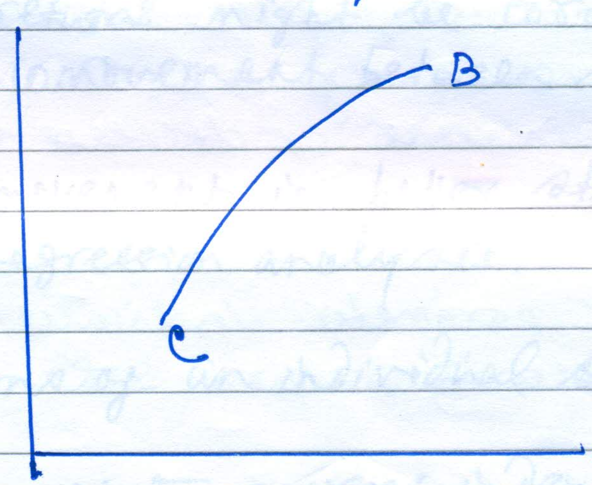
Now, we get 3 points namely B, E and C.

B gives highest return
C is the global minimum risk portfolio, and
E lies in between
when, we join these three points, we get efficient frontier.

So, to conclude;

The set of portfolios lying between the global minimum risk portfolio and the maximum return portfolio on the efficient frontier represents the efficient set of portfolios.

The efficient frontier is a concave curve in the risk - return space.



Limitations of Markowitz model

i) Large number of input data required for calculations:
like; * estimate of returns of each security
* st. deviation of returns of each security
* covariances of returns for each pair of
* securities in the portfolio.

Total data required : $2N + N(N-1)/2$

Eg: For analysing a set of 200 securities, we require 200 return estimates 200 variance estimates and 19900 covariance estimates making a total of 20300 estimates.

(ii) Complexities of computations.

(iii) Little use in practical applications.

Index Models William Sharpe

Basic notions:

1. All stocks are affected by movements in the stock market.

2. It is seen, when market moves up, prices of most shares tend to increase. and vice-versa

3. Security returns might be correlated and there is comovement between securities.

4. This co-movement is better studied through Linear Regression analysis.

Returns of an individual securities = R_i

Returns of the market index = R_m

Equation may be:

residual return

$$R_i = \alpha_i + \beta_i R_m + e_i$$

This equation has 2 parts

α_i = independent of market

β_i = dependent and due to market.

β measures the sensitivity of stock return to the return on the market index.

Aap mujhe achhe lagne lage.....
Ei mere kasam li....eitne kasam li

It implies;

i) if $\beta_i = 2$, then if market return (R_m) increases by 10%, then return of stock would go up by 20%.

ii) if market decreases by 10%, the return of stock would decrease by 20%.

iii) if $\beta_i = 0.5$, if market return increases or decreases by 10%, the security return would increase or decrease by 5%.

iv) The α parameter indicates what would be the security return when market return is '0'.

Eg, suppose $\alpha = 3$

(a) It indicates security return = 3, if market return is zero

(b) if α = ~~-4~~ -4.5, then security return would lose 4.5, if market return is '0' and when market return increases from '0', the security return would be less by 4.5.

(c) Positive α return is a bonus to a security.

Finding Security Return and Risk under Single index model.

Security Return: $R_i = \alpha_i + \beta_i R_m$

α_i = Return component specific to the security

$\beta_i R_m$: a market related return

Risk of Security: σ_i^2 would comprise 2 factors : market related risk + specific risk

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$$

Eg: The estimated values α_i, β_i and σ_{ei}^2 of a security are 2%, 1.5 and 300 respectively. The market index provides a return of 20% with a variance of 120, find risk and return.

Ans Return : $R_i = \alpha_i + \beta_i R_m$

$= 2 + 1.5(20)$

$= 32\ \%$

$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$

$= (1.5)^2(120) + 300$

$= \cancel{22.5}\ 2.25(120) + 300$

$= 270 + 300 = 570$

$\therefore \sigma_i = \sqrt{570} = 23.87\ \%$

Portfolio Return and Risk

Return of portfolio $= R_p = \alpha_p + \beta_p R_m$

we get; $\alpha_p = \sum \alpha_i \omega_i$

$$\beta_p = \sum \beta_i \omega_i$$

ω_i = proportion of investment in i^{th} security

Risk of Portfolio; σ_p

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{n} \omega_i^2 \sigma_{ei}^2$$

When, there are more number of securities in a portfolio, the $[\omega_i^2 \sigma_{ei}^2]$ becomes zero or quite negligible.

So, $\sigma_p = \sqrt{\beta_p^2 \sigma_m^2} = \beta_p \sigma_m$

Q Eg: Input data

Security	Weightage	α	β	σ_{ei}^2
A	0.2	2.0	1.7	370
B	0.1	3.5	0.5	240
C	0.4	1.5	0.7	410
D	0.3	0.75	1.3	285
Portfolio value	1.0	1.575	1.06	108.45

market Return = 15%
market variance = 320

Ans. How do we get α_p, β_p, σ_{ei}^2?

$$\alpha_p = \sum \alpha_i w_i$$
$$= (0.2)(2.0) + (0.1)(3.5) + (0.4)(1.5) + (0.3)(0.75)$$
$$= 1.575$$

$$\beta_p = \sum \beta_i w_i$$
$$= (1.7)(0.2) + (0.5)(0.1) + (0.7)(0.4) + (1.3)(0.3)$$
$$= 1.06$$

$$\sigma_{ei}^2 = \sum \sigma_e^2 w_i$$
$$= (370)(0.2) + (240)(0.1) + (410)(0.4) + (285)(0.3)$$
$$= 108.45$$

Now, $R_p = \alpha_p + \beta_p R_m$
$$= 1.575 + (1.06)(15)$$
$$= 17.475$$

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum \omega_i^2 \sigma_{ei}^2$$

$$= (1.06)^2 (320) + 108.45$$

$$= 468.002$$

$$\therefore \sigma_p = \sqrt{468}$$

$$= 21.63$$

<u>Home task</u>

Security	Weight	α_i	β_i	Residual Variance σ_{ei}^2
1	0.2	2.0	1.2	320
2	0.3	1.7	0.8	450
3	0.1	−0.8	1.6	270
4	0.4	1.2	1.3	180

Return on market index R_m = 16.4 %
Standard devn of return of R_m = 14 %

Calculate Return and Risk of Portfolio

<u>Ans</u> $R_p = 20.334$ %

$\sigma_p = 18.67$ %

~~Multi index model~~

Q following data are given.

Stock	α_i	β_i	Residual variance σ_{ei}^2
1	-2.1	1.6	14
2	1.8	0.4	8
3	1.2	1.3	18

which single stock an investor would prefer to own if the market index is 15% and variance of return is 20%.

Ans: Here, we have to calculate the expected return and risk of each security under single index model.

Return $R_i = \alpha_i + \beta_i R_m$

Stock-1 $R_1 = -2.1 + (1.6)(15) = -2.1 + 24 = 21.9$

2 $R_2 = 1.8 + (0.4)(15) = 1.8 + 6 = 7.8$

3 $R_3 = 1.2 + (1.3)(15) = 1.2 + 19.5 = 20.7$

Risk $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$

for Stock 1 $= (1.6)^2(20) + 14 = 51.2 + 14 = 65.2$

2 $= (0.4)^2(20) + 8 = 3.2 + 8 = 11.2$

3 $= (1.3)^2(20) + 18 = 33.8 + 18 = 51.8$

$\sigma_1 = \sqrt{65.2} = 8.07$ $\quad\sigma_3 = \sqrt{51.8} = 7.2$

$\sigma_2 = \sqrt{11.2} = 3.35$

Now, we have to compute return per unit of risk for each stock.

$$\text{Stock-1} = \frac{\text{Return}}{\text{Risk}} = \frac{21.9}{8.07} = 2.71$$

$$\text{Stock-2} = \frac{7.8}{3.35} = 2.33$$

$$\text{Stock-3} = \frac{20.7}{7.3} = 2.84$$

It is seen stock-3 shows highest return per every unit of risk, so we may recommend to invest in stock-3.

Q How many parameters must be estimated to analyse the risk-return portfolio of a 50 stock portfolio using (a) Markowitz model and (b) Sharpe single index model?

Ans Markowitz model : we require
- N return estimates
- N variance estimates
- $N(N-1)/2$ covariance estimates

Total Estimates : $2N + N(N-1)/2 = 2\times50 + 50(49/2)$
$= 1325$

Sharpe single estimates : N α estimates
N β estimates
N Residual variance estimates
Market Return (R_m)
Variance of market return $(\sigma_m)^2$

Total Estimates
$3N + 2$
$= 3\times50 + 2$
$= 152$ Ans

Multi Index model.

Takes care of additional factors relating to market such as inflation, real economic growth, interest rates and exchange rates.

In that case, the formula shall be

$$R_i = \alpha_i + \beta_m R_m + \beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 + e_i$$